Solution to Test 2, MMAT5000

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1. **Solution:** If we take the limits through the path $x = y^2$, then

$$\lim_{y\to 0} f(y^2, y) = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \frac{1}{2},$$

So, if $f(x,y)$ has a limit as $(x,y) \to (0,0)$, it must be $\frac{1}{2}$. However, for $\varepsilon = \frac{1}{11}$, for $\forall \delta \in (0, \sqrt{3})$, there exists $(x,y) = (\frac{1}{2}\delta^2, \frac{1}{2}\delta)$ with $\|(\frac{1}{2}\delta^2, \frac{1}{2}\delta) - (0,0)\| = \frac{\sqrt{\delta^2+1}}{2}\delta < \delta$ such that

$$\begin{aligned}\left|\frac{xy^2}{x^2+y^4} - \frac{1}{2}\right| &= \left|\frac{\frac{1}{2}\delta^2 \cdot (\frac{1}{2}\delta)^2}{(\frac{1}{2}\delta^2)^2 + (\frac{1}{2}\delta)^4} - \frac{1}{2}\right| \\ &= |\frac{2}{5} - \frac{1}{2}| \\ &= \frac{1}{10} \\ &> \varepsilon.\end{aligned}$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2. **Solution:**

(i) Let (x_0, y_0) be any point in $\mathbb{R}^2$.

For any $\varepsilon > 0$, there exists $\delta \in \left(0, \min\{1, \frac{\varepsilon}{2|x_0|+1+2|x_0|^2(|y_0|+1)}\}\right)$ such that for any $\|(x,y) - (x_0,y_0)\| < \delta$,

$$\begin{aligned}|g(x,y) - g(x_0,y_0)| &= \left|x^2\sin(x+y^2) - x_0^2\sin(x_0+y_0^2)\right| \\ &= \left|x^2\sin(x+y^2) - x_0^2\sin(x+y^2) + x_0^2\sin(x+y^2) - x_0^2\sin(x_0+y_0^2)\right| \\ &\le \left|x^2\sin(x+y^2) - x_0^2\sin(x+y^2)\right| + \left|x_0^2\sin(x+y^2) - x_0^2\sin(x_0+y_0^2)\right| \\ &= |(x^2-x_0^2)\sin(x+y^2)| + \left|x_0^2(\sin(x+y^2) - \sin(x_0+y_0^2))\right| \\ &= |(x^2-x_0^2)\sin(x+y^2)| + \left|2x_0^2\cos\frac{x+x_0+y^2+y_0^2}{2}\sin\frac{x-x_0+y^2-y_0^2}{2}\right| \\ &\le |x^2-x_0^2| + \left|2x_0^2\sin\frac{x-x_0+y^2-y_0^2}{2}\right| \\ &= |x+x_0||x-x_0| + |x_0|^2|x-x_0+y^2-y_0^2| \\ &\le |x+x_0||x-x_0| + |x_0|^2(|x-x_0| + |y+y_0||y-y_0|) \\ &\le (2|x_0|+\delta)\delta + |x_0|^2(\delta + (2|y_0|+\delta)\delta) \\ &\le [2|x_0|+1+2|x_0|^2(|y_0|+1)]\delta \\ &\le \varepsilon.\end{aligned}$$

Here we used the fact that

$$|\sin t| \le |t|, \quad t \in \mathbb{R}$$

$$|x + x_0| \le |x - x_0| + |2x_0| \le 2|x_0| + \delta,$$

$$|y + y_0| \le |y - y_0| + |2y_0| \le 2|y_0| + \delta.$$

Therefore, $g(x, y)$ is continuous at any point $(x_0, y_0) \in \mathbb{R}^2$, i.e. $g(x, y)$ is continuous on $\mathbb{R}^2$.

(ii) If we take $(x_0, y_0) = (-1, 2)$, and $\varepsilon = 0.1$, then we can take $\delta = 0.01$.

3.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^2}{5n^2 + n} \\ &= \lim_{n\to\infty} \frac{1}{5 + n^{-1}} \\ &= \frac{1}{5} \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{-k} \\ &= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{e^{-1}(1 - (e^{-1})^n)}{1 - e^{-1}} \\ &= 0. \end{aligned}$$

Therefore, $\lim_{n\to\infty} (a_n, b_n) = (\frac{1}{5}, 0)$.

4. **Proof:** For any $\varepsilon > 0$, there exists $\delta = \frac{1}{2}$, such that for any $(x, y) \in X$ satisfying $\rho((x, y) - (0, 0)) < \delta$, we have

$$\tau(h(x, y) - h(0, 0)) = |h(x, y) - h(0, 0)| = |h(0, 0) - h(0, 0)| = 0 < \varepsilon.$$

Therefore, $h(x, y)$ is continuous at $(0, 0)$ with respect to $\rho - \tau$.